Technical Notes

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van Leer Flux Vector Splitting in Moving Coordinates

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Introduction

A SIMPLE method of incorporating upwinding into a finite-difference scheme for the solution of a system of hyperbolic conservation laws is to represent the physical flux vector $F(w)$, where w is the vector of conserved variables, as the sum of split fluxes $F^+(w)$ and $F^-(w)$ such that the Jacobian matrices dF^+/dw and dF^-/dw have no negative and positive eigenvalues, respectively. Each split-flux vector is then differenced using forward or backward differences according to the sign of the eigenvalues of the associated Jacobian matrix. One such scheme for the Euler equations of gas dynamics is the split-flux method of van Leer.[1] In Ref. 1, the construction of the split-flux vectors is referred to a three-dimensional Cartesian coordinate system. This construction has been generalized by Anderson et al.[2] and Thomas et al.[3] for applications in which curvilinear (body-conforming) coordinate systems are used. In the present study, we derive the split-flux vectors referred to moving curvilinear coordinates. This further generalization makes it possible to apply the method to problems in which it might be convenient or necessary to use moving grids. Situations of this type would be encountered, for example, in helicopter rotor calculations, in which the grid is attached to a blade which, in general, undergoes translation, rotation, and deformation. The time-accurate calculation of flows on solution-adaptive grids is another instance in which it is necessary to allow for arbitrary grid-point motion.

van Leer Flux Vector Splitting

The particular choice of split fluxes proposed by van Leer[1] was made under the following restrictions:

1) $F^\pm$ must be continuous, with $F^+(w)=F(w)$ for Mach numbers $M \geq 1$ and $F^-(w)=F(w)$ for Mach numbers $M \leq -1$,

2) The components $F_i^\pm$ of $F^\pm$ must satisfy $F_i^+(M) = \pm F_i^-(-M)$ if $F_i(M) = \pm F_i(-M)$,

3) $dF^\pm/dw$ must be continuous,

4) $dF^\pm/dw$ must have one eigenvalue vanish for $|M|<1$, and

5) $F^\pm(M)$, like $F(M)$, must be a polynomial in M of the lowest possible degree.

The justification for the choice of restrictions 1-5 is given in Ref. 1 and will not be repeated here.

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One-Dimensional Splitting

For completeness and future reference, we present the full- and split-flux vectors for the one-dimensional Euler equations. The full-flux, written as a function of the density ρ, the speed of sound a, and the Mach number, is

$$F(\rho,a,M) = \begin{bmatrix} \rho a M \\ \rho a^2\left(M^2+\dfrac{1}{\gamma}\right) \\ \rho a^3 M\left(\dfrac{1}{2}M^2+\dfrac{1}{\gamma-1}\right) \end{bmatrix} \tag{1}$$

where γ is the ratio of specific heats. The split-flux vector, derived in Ref. 1, is

$$F^\pm(\rho,a,M) = \begin{bmatrix} F_m^\pm = \pm\dfrac{\rho a}{4}(M\pm 1)^2 \\ F_M^\pm \equiv \dfrac{aF_m^\pm}{\gamma}[(\gamma-1)M\pm 2] \\ \dfrac{\gamma^2}{2(\gamma^2-1)}(F_M^\pm)^2/F_m^\pm \end{bmatrix}, \quad |M|<1 \tag{2}$$

We now derive the split fluxes for a coordinate system moving with speed ξ_t. In this case, the full-flux vector reads

$$F = \begin{bmatrix} \rho a\bar{M} \\ \rho a^2\left[\dfrac{1}{\gamma}+\left(\bar{M}-\dfrac{\xi_t}{a}\right)\bar{M}\right] \\ \rho a^3\left[\left(\dfrac{1}{\gamma-1}+\dfrac{1}{2}\left(\bar{M}-\dfrac{\xi_t}{a}\right)^2\right)\bar{M}-\dfrac{\xi_t}{\gamma a}\right] \end{bmatrix} \tag{3}$$

where

$$\bar{M} = M + \frac{\xi_t}{a}$$

is the contravariant Mach number.

Conditions 1 and 3 require that the components $F_i^\pm$ of the split-flux vector $F^\pm$ satisfy

$$F_i^\pm(\bar{M}=\pm 1) = F_i(\bar{M}=\pm 1), \qquad F_i^\pm(\bar{M}=\mp 1)=0 \tag{4}$$

$$\left.\frac{\partial F_i^\pm}{\partial \bar{M}}\right)_{\bar{M}=\pm 1} = \left.\frac{\partial F_i}{\partial \bar{M}}\right)_{\bar{M}=\pm 1}, \qquad \left.\frac{\partial F_i^\pm}{\partial \bar{M}}\right)_{\bar{M}=\mp 1} = 0 \tag{5}$$

The split mass flux vector $F_m^\pm$ follows directly from Eqs. (4) and (5); we have

$$F_m^\pm = \pm\frac{\rho a}{4}(\bar{M}\pm 1)^2 \tag{6}$$

which satisfies conditions 2 and 5. The split momentum flux

$F_m^\pm$ can also be deduced from Eqs. (4) and (5) using Eq. (2) as a guide:

$$F_{\bar M}^\pm \equiv \frac{aF_m^\pm}{\gamma}\left[(-\bar M \pm 2) + \gamma\left(\bar M - \frac{\xi_t}{a}\right)\right] \tag{7}$$

which again satisfies conditions 2 and 5.

The splitting for the energy flux is somewhat more complicated. From condition 5 we postulate that the split energy flux can be written as

$$F_E^\pm = c_0^\pm + c_1^\pm \bar M + c_2^\pm \bar M^2 + c_3^\pm \bar M^3 + c_4^\pm \bar M^4 \tag{8}$$

Equations (4) and (5) provide eight equations in the ten unknown coefficients $c_i^\pm$. A solution of these eight equations is

$$c_1^\pm = \frac{\rho a^3}{4}\left[\frac{2}{\gamma-1} + \left(1 \mp \frac{\xi_t}{a}\right)^2 - \left(1 \mp \frac{\xi_t}{a}\right) \mp \frac{3\xi_t}{a}\right] \tag{9}$$

$$c_2^\pm = \frac{\rho a^3}{4}\left[\pm 3\left(\frac{1}{\gamma-1} + \frac{1}{2}\left(1 \mp \frac{\xi_t}{a}\right)^2\right) \mp \left(1 \mp \frac{\xi_t}{a}\right) - \frac{4\xi_t}{\gamma a}\right] - 2c_0^\pm \tag{10}$$

$$c_3^\pm = \frac{\rho a^3}{4}\left[\left(1 \mp \frac{\xi_t}{a}\right) \pm \frac{\xi_t}{\gamma a}\right] \tag{11}$$

$$c_4^\pm = \frac{\rho a^3}{4}\left[\mp\left(\frac{1}{\gamma-1} + \frac{1}{2}\left(1 \mp \frac{\xi_t}{a}\right)^2\right) \pm \left(1 \mp \frac{\xi_t}{a}\right) + \frac{2\xi_t}{\gamma a}\right] + c_0^\pm \tag{12}$$

The undetermined coefficients $c_0^\pm$ must be chosen such that the sum $c_0^+ + c_0^-$ is equal to the leading term in the polynomial for full energy flux in Eq. (3) and the Jacobian matrices $dF^\pm/dw$ have the required properties. The procedure used here is based on analogy with the form of the split energy flux in Eq. (2). Using Eqs. (6) and (7) to form the combination of split fluxes in Eq. (2), we have

$$\frac{\gamma^2}{2(\gamma^2-1)}\frac{(F_{\bar M}^\pm)^2}{F_m^\pm} = \pm\frac{\rho a^3}{8(\gamma^2-1)}(\bar M^2 \pm 2\bar M + 1) \times\left[(\gamma-1)^2\bar M^2 \pm 4(\gamma-1)\bar M - 2\gamma(\gamma-1)\bar M\frac{\xi_t}{a} \mp 4\gamma\frac{\xi_t}{a} + \gamma^2\frac{\xi_t^2}{a^2} + 4\right] \tag{13}$$

We now compare the coefficients of Eq. (13) with Eqs. (9-12) obtained earlier. From Eq. (13) we have

$$c_1^\pm = \frac{\rho a^3}{4}\left[\frac{2}{\gamma-1} \mp \frac{\gamma(\gamma+3)}{\gamma^2-1}\frac{\xi_t}{a} + \frac{\gamma^2}{\gamma^2-1}\frac{\xi_t^2}{a^2}\right] \tag{14}$$

and

$$c_3^\pm = \frac{\rho a^3}{4}\left[1 \mp \frac{\gamma}{\gamma+1}\frac{\xi_t}{a}\right] \tag{15}$$

Comparing Eq. (14) to Eq. (9) and Eq. (15) to Eq. (11) suggests that the coefficients of ξ_t and ξ_t^2 in Eq. (13) must be multiplied by the factor $(\gamma^2-1)/\gamma^2$. Equation (13), so modified, gives the required splitting:

$$F_E^\pm = \frac{\gamma^2}{2(\gamma^2-1)}\left[\frac{(F_{\bar M}^\pm)^2}{F_m^\pm} + \frac{2}{\gamma^2}\xi_t F_{\bar M}^\pm + \frac{1}{\gamma^2}\xi_t^2 F_m^\pm\right] \tag{16}$$

Note that the degeneracy of $F_E^\pm$ ensures that condition 4 is met. The nonzero eigenvalues λ^+ of dF^+/dw are the roots of quadratic equation

$$(\lambda^+)^2 - \frac{3}{2}a\lambda^+(1+\bar M)\left[1 - \frac{\gamma-1}{12\gamma(\gamma+1)}(\bar M-1)[\gamma(\bar M-1)^2 + 2\gamma(\bar M-1) - 2(\gamma+3)]\right] + \frac{a^2}{4}(1+\bar M)^3\left[1 - \frac{\bar M-1}{8\gamma(\gamma+1)}[4\gamma(\gamma-1)(\bar M-1) + (\gamma+1)(3-\gamma)]\right] = 0, \quad |\bar M| < 1 \tag{17}$$

In the range $1 \le \gamma \le 3$, the roots of Eq. (17) are positive.[1] By symmetry, it follows that the nonzero roots of the corresponding equation for λ^- are negative.

Multidimensional Splitting in Cartesian Coordinates

The splitting for multidimensional flow in Cartesian coordinates is easily deduced from that for one-dimensional flow. The split-flux vectors in the three coordinate directions are of similar form; therefore, only the split-flux vector in the x direction is shown here. We have

$$F^\pm = \begin{Bmatrix} \pm\frac{\rho a}{4}(\bar M_x \pm 1)^2 = F_m^\pm \\ \frac{a}{\gamma}F_m^\pm[(-\bar M_x \pm 2) + \gamma M_x] \neq \mathrm{N}_{\bar M_x}^\pm \\ aF_m^\pm M_y \\ aF_m^\pm M_z \\ F_E^\pm \end{Bmatrix} \tag{18}$$

where

$$\bar M_x = \frac{u + \xi_t}{a} = M_x + \frac{\xi_t}{a}, \qquad M_y = \frac{v}{a}, \qquad M_z = \frac{w}{a}$$

u, v, and w are the Cartesian components of the fluid velocity, and

$$F_E^\pm = \frac{\gamma^2}{2(\gamma^2-1)}\left[\frac{(F_{\bar M_x}^\pm)^2}{F_m^\pm} + \frac{2}{\gamma^2}\xi_t F_{\bar M_x}^\pm + \frac{1}{\gamma^2}\xi_t^2 F_m^\pm\right] + \frac{F_m^\pm}{2}(v^2 + w^2)$$

Splitting in Curvilinear Coordinates

It is often convenient to write the governing equations in a moving, body-fitted coordinate system. This is accomplished by the transformation

$$\xi = \xi(x,y,z,t), \qquad \eta = \eta(x,y,z,t)$$
$$\zeta = \zeta(x,y,z,t), \qquad \tau = t \tag{19}$$

We denote the Jacobian of the transformation, Eq. (19) by J.

The transformed governing equations in strong conservation form may be written as

$$\frac{\partial \bar Q}{\partial \tau} + \frac{\partial \bar F}{\partial \xi} + \frac{\partial \bar G}{\partial \eta} + \frac{\partial \bar H}{\partial \zeta} = 0 \tag{20}$$

Following Anderson et al.,[2] Eq. (20) is transformed by a local rotation matrix T so that the flux vector $\bar{F}$ can be split in a one-dimensional fashion in a direction along a ξ = constant line, treating the $\partial\bar{G}/\partial\eta$ and $\partial\bar{H}/\partial\zeta$ as source terms. It is shown in Ref. 2 that the transformed flux vector

$$\hat{F} = T\bar{F}$$

is of the same form as the Cartesian flux vector and can thus be split according to the scheme developed for Cartesian coordinates after replacing the Cartesian velocity components by the corresponding rotated velocity components. The inverse transformation is then applied to the split fluxes $\hat{F}^{\pm}$ to obtain the splitting for $\bar{F}$. The split fluxes $\bar{F}^{\pm}$ are

$$\bar{F}^{\pm} = \frac{|\nabla\xi|}{J} \begin{Bmatrix} \bar{F}^{\pm}_{m} \\ \frac{a}{\gamma}\bar{F}^{\pm}_{m}\left[\frac{\xi_x}{|\nabla\xi|}(-\bar{M}_{\xi} \pm 2) + \gamma M_x\right] \\ \frac{a}{\gamma}\bar{F}^{\pm}_{m}\left[\frac{\xi_y}{|\nabla\xi|}(-\bar{M}_{\xi} \pm 2) + \gamma M_y\right] \\ \frac{a}{\gamma}\bar{F}^{\pm}_{m}\left[\frac{\xi_z}{|\nabla\xi|}(-\bar{M}_{\xi} \pm 2) + \gamma M_z\right] \\ \bar{F}^{\pm}_{E} \end{Bmatrix} \tag{21}$$

where

$$\bar{F}^{\pm}_{m} = \pm\frac{\rho a}{4}(\bar{M}_{\xi} \pm 1)^2$$

$$\bar{F}^{\pm}_{E} = \frac{\gamma^2}{2(\gamma^2-1)}\left[\frac{(\bar{F}^{\pm}_{M_\xi})^2}{\bar{F}^{\pm}_{m}} + \frac{2}{\gamma^2}\frac{\xi_t}{|\nabla\xi|}\bar{F}^{\pm}_{M_\xi} + \frac{1}{\gamma^2}\frac{\xi_t^2}{|\nabla\xi|^2}\bar{F}^{\pm}_{m}\right] + \frac{\bar{F}^{\pm}_{m}}{2}(\bar{v}^2 + \bar{w}^2)$$

$$\bar{F}^{\pm}_{M_\xi} = \frac{a}{\gamma}\bar{F}^{\pm}_{m}[(-\bar{M}_{\xi} \pm 2) + \gamma M_{\xi}]$$

$$a\bar{M}_{\xi} = (u\xi_x + v\xi_y + w\xi_z + \xi_t)/|\nabla\xi| = \bar{u} + \xi_t/|\nabla\xi|$$

$$aM_{\xi} = \bar{u}$$

Note also that

$$u^2 + v^2 + w^2 = \bar{u}^2 + \bar{v}^2 + \bar{w}^2$$

where $\bar{u}$, $\bar{v}$, and $\bar{w}$ are the rotated velocity components. The expression corresponding to Eq. (21) for stationary curvilinear coordinates is presented by Thomas et al.[3] The split fluxes $\bar{G}^{\pm}$ and $\bar{H}^{\pm}$ are obtained from Eq. (21) by replacing ξ by η and ζ, respectively.

Concluding Remarks

In this Note, the van Leer split-flux vectors have been derived for moving curvilinear coordinate systems. The split fluxes presented reduce to those given by van Leer[1] and Thomas et al.[3] for the case of stationary coordinates.

The split-flux vectors obtained in the present study have been successfully applied to a fixed-wing calculation in which the relative motion between the wing and the fluid was introduced through the grid motion. The application of the result obtained here to the calculation of helicopter rotor flowfields is currently being investigated.

References

[1]van Leer, B., "Flux-Vector Splitting for the Euler Equations," *Lecture Notes in Physics*, Vol. 170, Springer-Verlag, New York, 1982, pp. 507-512.

[2]Anderson, W. K., Thomas, J. L., and van Leer, B., "A Comparison of Finite Volume Flux Vector Splittings for the Euler Equations," AIAA Paper 85-0122, Reno, NV, 1985.

[3]Thomas, J. L., van Leer, B., and Walters, R. W., "Implicit Flux-Split Schemes for the Euler Equations," AIAA Paper 85-1680, Cincinnati, OH, 1985.

Turbulent Near Wake of a Symmetrical Body

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Nomenclature

C_a	= Crocco number of adjacent flow
k	= ratio of specific heats
M_a	= Mach number of adjacent flow
u	= velocity
u_c	= centerline velocity
u_a	= velocity adjacent to wake region
u_{max}	= maximum value of adjacent velocity if all the energy of the flow is in the form of kinetic energy
r	= radius
R	= radius where adjacent flow begins
y	= two-dimensional coordinate normal to flow direction
Y	= y where adjacent flow begins
ρ	= density
ρ_a	= adjacent flow density

Introduction

When a symmetrical body is immersed in a uniform flow at zero angle of attack, an interesting conclusion can be drawn about the near wake profile. The conclusion is limited to those wakes in which vortex shedding is not present or is negligible.

Formulation

The location of the near wake profile is defined to be downstream of the body where the pressure gradient normal to the centerline has vanished and the time averaged streamlines have become straight and parallel. If we assume that dominant vortex shedding is not present, the time averaged values of the velocity in the near wake profile can be approximately represented by a cosine function as shown in Fig. 1. Thus, in the near wake

$$\frac{u}{u_a} = \frac{u_c}{u_a} + (0.5)\left(1 - \frac{u_c}{u_a}\right)\left(1 - \cos\left[\frac{r}{R}\,180\text{ deg}\right]\right) \tag{1}$$

where $r \to y$ and $R \to Y$ for the two-dimensional case.

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